Combinatorics of binomial decompositions of the simplest Hodge integrals

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Abstract. We reduce the calculation of the simplest Hodge integrals to some sums over decorated trees. Since Hodge integrals are already calculated, this gives a proof of a rather interesting combinatorial theorem and a new representation of Bernoulli numbers.

1. Introduction

In this paper we study the simplest Hodge integrals on the moduli space of curves. Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of curves of genus g with n marked points. By ψ_i denote the first Chern class of the line bundle over $\overline{\mathcal{M}}_{g,n}$, whose fiber at a moduli point $(C_g, x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g,n}$ is equal to $T^*_{x_i}C$. The Hodge bundle is the rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$, whose fiber at a moduli point $(C_g, x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g,n}$ is equal to $H^0(C, \omega_C)$. By $\lambda_1, \dots, \lambda_g$ denote the Chern classes of the Hodge bundle.

In this paper we consider the integrals

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2-i}\lambda_i. \tag{1.1}$$

We give an algorithm to calculate these integrals. In fact, these integrals have already been calculated by Faber and Pandharipande, see [**2**]. There is a remarkable formula for the generating function of Hodge integrals:

$$1 + \sum_{g\geq 1}\sum_{i=0}^{g} t^{2g}k^i \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2+i}\lambda_{g-i} = \left(\frac{t/2}{\sin(t/2)}\right)^{k+1}. \tag{1.2}$$

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So, let us explain our motivations. First, for the integrals $\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2}\lambda_g$ our computing algorithm has a rather simple interpretation as a certain sum over trees. So, since these integrals are already calculated, we solve an interesting combinatorial problem. In particular, this gives a representation of Bernoulli numbers as some sums over trees.

Second, the combinatorial structure of Hodge integrals still seems to produce some questions, see [**3**]. Our approach could clarify something in this direction. Moreover, the soul of our computing algorithm is a 'cut-and-join' type equation for the Hodge integrals over two-pointed ramification cycles. Since Hurwitz numbers satisfy the similiar equations, this could give some new relations between Hurwitz numbers and Hodge integrals, see [**1, 4, 6**].

Third, our computing algorithm is a modification of the computing algorithm for the simplest Witten intersections, see [**7**]. It works very good to calculate any concrete intersection number, but it is very hard to prove any general statement using this approach. In the case considered in this paper we reduce the computing algorithm to rather friendly combinatorics. We hope this could help in our approach to the Witten conjecture.

Organization of the paper. In Section 2 we give our algorithm for computing the integrals $\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2}\lambda_g$. In Section 3 explain the combinatorial interpretation of this algorithm and combinatorial corollaries of this interpretation. In Section 4 we prove all theorems of Section 2.

In Appendix A we give some calculations checking independently our algorithm. In Appendix B we describe separately genus zero case of our combinatorial results. In Appendix C we generalize our algorithm to compute all Hodge integrals $\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2-i}\lambda_i$.

2. Calculation of Hodge integrals

In this section we explain an algorithm to calculate the Hodge integrals $\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2}\lambda_g$. The proofs of all theorems of this section are given in Section 4.

2.1. Some notations. Consider the moduli space of curves $\overline{\mathcal{M}}_{g,n+1}$. Let us fix some positive integers $a_1, \dots, a_n$. By $V_g^\circ(a_1, \dots, a_n)$ denote the subvariety of the open moduli space $\mathcal{M}_{g,n+1}$ consisting of curves $(C, x_1, \dots, x_{n+1})$ such that $-(\sum_{i=1}^n a_i)x_1 + a_1x_2 + \dots + a_nx_{n+1}$ is a divisor of a meromorphic function.

Let $V_g(a_1,\dots,a_n)$ be the closure of $V_g^\circ(a_1,\dots,a_n)$ in $\overline{\mathcal{M}}_{g,n}$. By $W_g(\prod_{i=1}^n \eta_{a_i})$ denote the integral of $\psi_1^{g+n-2}\lambda_g$ over the subspace $V_g(a_1,\dots,a_n)$:

$$W_g(\prod_{i=1}^n \eta_{a_i}) := \int_{V_g(a_1,\dots,a_n)} \psi_1^{g+n-2}\lambda_g. \tag{2.1}$$

2.2. Binomial decomposition.

THEOREM 2.1. *For arbitrary positive integers $a_1,\dots,a_n$ we have*

$$(-1)^g g! \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2}\lambda_g = \binom{g}{0} W_g(\prod_{i=1}^n \eta_{a_i}) - \binom{g}{1} W_g(\eta_1 \prod_{i=1}^n \eta_{a_i}) + \dots + (-1)^g \binom{g}{g} W_g(\eta_1^g \prod_{i=1}^n \eta_{a_i}). \tag{2.2}$$

2.3. Recursion relation. There is a recursion relation for the the numbers $W_g(\prod_{i=1}^n \eta_{a_i})$:

THEOREM 2.2. *If $g+n-2>0$, then*

$$(\sum_{i=1}^n a_i)(2g+n-1) W_g(\prod_{i=1}^n \eta_{a_i}) = \sum_{k<l} (a_k+a_l) W_g(\eta_{a_k+a_l} \prod_{i\neq k,l} \eta_{a_i}) + \sum_{k=1}^n \frac{a_k^3-a_k}{12} W_{g-1}(\prod_{i=1}^n \eta_{a_i}). \tag{2.3}$$

2.4. Initial values. Consider the case $g+n-2=0$. This means that either $g=1$, $n=1$, or $g=0$, $n=2$. In these cases we have the following:

THEOREM 2.3.

$$W_1(\eta_{a_1}) = \frac{a_1^2-1}{24}; \qquad W_0(\eta_{a_1}\eta_{a_2}) = 1. \tag{2.4}$$

2.5. Calculation of Hodge integrals. Now we have an algorithm to calculate the integrals $\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2}\lambda_g$. First, we express such integral via the numbers $W_g(\prod_{i=1}^n \eta_{a_i})$ (Theorem 2.1). Then we step by step simplify the numbers $W_g(\prod_{i=1}^n \eta_{a_i})$ using Theorem 2.2 until we obtain numbers calculated in Theorem 2.3.

Let us apply this algorithm to calculate $\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1$. The first step looks, for example, like follows:

$$-\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = W_1(\eta_2) - W_1(\eta_1\eta_2).$$

Then

$$3 \cdot 3 \cdot W_1(\eta_1\eta_2) = 3 \cdot W_1(\eta_3) + \frac{8-2}{12} W_0(\eta_1\eta_2).$$

Since

$$W_0(\eta_1\eta_2) = 1, \quad W_1(\eta_3) = \frac{8}{24}, \quad \text{and} \quad W_1(\eta_2) = \frac{3}{24},$$

we have

$$-\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = \frac{3}{24} - \frac{1}{3} \cdot \frac{8}{24} - \frac{1}{9} \cdot \frac{6}{12} = -\frac{1}{24}.$$

The same number is give by Equation 1.2.

3. The number $W_g(\eta_1^n)$ as a sum over trees

The algorithm of calculation of numbers $W_g(\eta_1^n)$ implies a representation of these numbers as a sum over some trees. In this section we describe the trees we need and prove a theorem expressing $W_g(\eta_1^n)$ via some numbers calculated by trees.

3.1. Decorated trees. Let us fix $g \geq 0$ and $n \geq 1$. We describe here trees corresponding to the number $W_g(\eta_1^n)$.

3.1.1. *Construction.* We consider rooted trees. Each vertex has ≤ 2 sons. [1] We require that there are exactly n vertices with no sons and exactly g vertices with one son. This follows that there are exactly $n-1$ vertices with two sons.

By leaves we call vertices with no sons. By V_0 denote the set of leaves. Let us assign to each leaf a personal number nm from the set $\{1, \dots, n\}$. In other words, we consider a one-to-one correspondence $\mathsf{nm} : V_0 \to \{1, \dots, n\}$.

One more condition for the structure of trees is that the father of each leaf has two sons (In fact, we use this condition just to decrease the number of examples. The contribution of such trees to our formulas will be zero).

By V_1 denote the set of vertices with one son. By V_2 denote the set of vertices with two sons. Consider a map $\mathsf{cp} : V_1 \cup V_2 \to \{1, \dots, 2g+n-1\}$ satifying the following three properties:

(1) The map cp takes different verices to different numbers.

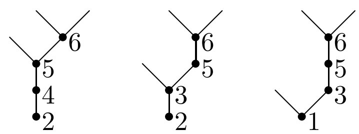
[1] If we have two connected vetices in a rooted tree, then one of these vertices is the son of another one. By son we call the vertex, which is farther from the root.

(2) If $a \in \mathsf{cp}\,(V_1)$, then $a > 1$ and $a - 1 \notin \mathsf{cp}\,(V_1 \cup V_2)$.
(3) If vertex v is a 'descendant' of vertex v', then $\mathsf{cp}\,(v) > \mathsf{cp}\,(v')$.

In particular, if the root vertex has one (resp., two) sons, then its image under the mapping cp is equal to 2 (resp., 1).

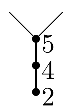
By a decorated tree (or (n, g)-decorated tree) we call all this data, i. e. a rooted tree with mappings nm and cp.

3.1.2. *Examples.* Consider $n = 3$, $g = 2$. Up to isomorphism, there are 3 possible rooted trees with the mapping cp.



Note that in this particular case the type of rooted graph uniquely determines the mapping cp. For each of these graphs there are three possible mappings nm.

Consider $n = 2$, $g = 2$. There is a unique possible rooted tree with the mapping cp. For this tree there is a unique possible mapping nm.

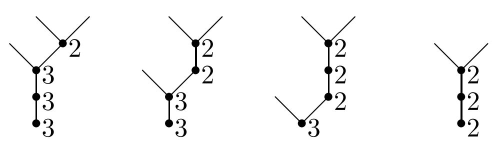


Consider $n = 1$, $g = 2$. There exists no such decorated trees. Moreover, if $n = 1$, then a decorated tree exists if and only if $g = 0$. In this case this tree consists just of one vertex.

3.2. Taking a number from a decorated tree.

3.2.1. *Function ml.* Consider a decorated tree. Let us define one more function $\mathsf{ml} : V_1 \cup V_2 \to \{1, \ldots, n\}$. The mapping ml takes a vertex to the number of its descendants in V_0.

In the examples studied above the mapping ml looks like follows:



3.2.2. *The numbers $N(\Gamma)$ and $S_{g,n}$.* To any (n, g)-decorated tree Γ we assign the number $N(\Gamma)$:

$$N(\Gamma) = \frac{1}{n^{(n+g-1)}} \cdot \prod_{v \in V_2} \frac{\mathsf{ml}\,(v)}{\mathsf{cp}\,(v)} \cdot \prod_{v \in V_1} \frac{\mathsf{ml}\,(v)^3 - \mathsf{ml}\,(v)}{12\mathsf{cp}\,(v)} \tag{3.1}$$

By $S_{g,n}$ denote the sum

$$S_{g,n} = \sum_{\Gamma} N(\Gamma), \tag{3.2}$$

where the sum is taken over all (n,g)-decorated trees Γ.

3.3. Theorems.

THEOREM 3.1. *For any $n \geq 1$, $g \geq 0$, we have*

$$W_g(\eta_1^n) = S_{g,n}. \tag{3.3}$$

This theorem implies purely combinatorial identities.

COROLLARY 3.2. *For any $g \geq 1$, $n \geq 1$, we have*

$$\binom{g}{0} S_{g,n+g} - \binom{g}{1} S_{g,n+g-1} + \cdots + (-1)^g \binom{g}{g} S_{g,n} = \frac{(2^{2g-1}-1)g!}{2^{2g-1}(2g)!}|B_{2g}|. \tag{3.4}$$

COROLLARY 3.3. *For any $n \geq 1$, we have*

$$S_{0,n} = 1. \tag{3.5}$$

Note that the trees determining the numbers $S_{0,n}$ have much more simple description then in the general case. So Corollary 3.3 has more simple and more beautiful reformulation.

3.4. Proof of Theorem 3.1. In fact, this Theorem is more or less obvious. Let us consider the calculation of the number $W_g\eta_1^n$ step by step. Simultaniously we try to build the corresponding graph.

At the beginning we have only n vertices $v_1^0, \ldots, v_n^0$. The first step given by Equation 2.2:

$$W_g\eta_1^n = \frac{1}{n(2g+n-1)} \sum_{i<j} 2W_g(\eta_2\eta_1^{n-2}). \tag{3.6}$$

Consider the summand in the right hand side of this equation corresponding to the pair (i,j), $i<j$. According to this summand we add to our graph the vertex v_{2g+n-1}^2 with two sons, v_i^0 and v_j^0. We put $\mathsf{cp}\,(v_{2g+n-1}^2) = 2g+n-1$ and we have $\mathsf{ml}\,(v_{2g+n-1}^2) = 2$. Therefore, the factor $2/(2g+n-1)$ is equal to $\mathsf{ml}\,(v_{2g+n-1}^2)/\mathsf{cp}\,(v_{2g+n-1}^2)$ (Let us always skip the factor $1/n$ throughout our calculations).

Let us make the second step for this summand. We have:

$$W_g(\eta_2\eta_1^{n-2}) = \frac{1}{n(2g+n-2)}\sum_{i'<j'} 2W_g(\eta_2^2\eta_1^{n-4}) + \frac{1}{n(2g+n-2)}\sum_{i'} 3W_g(\eta_3\eta_1^{n-3}) + \frac{1}{n(2g+n-2)}\cdot\frac{2^3-2}{12}\cdot W_{g-1}(\eta_2\eta_1^{n-2}). \tag{3.7}$$

Here we take all sums over $i', j' \in \{1,\dots,n\}\setminus\{i,j\}$.

Consider the summand of the first sum in the right hand side of this equation corresponding to the pair (i',j'), $i'<j'$. According to this summand we add to our graph the vertex v^2_{2g+n-2} with two sons, $v^0_{i'}$ and $v^0_{j'}$. We put $\mathsf{cp}\,(v^2_{2g+n-2}) = 2g+n-2$ and we have $\mathsf{ml}\,(v^2_{2g+n-2}) = 2$. Therefore, the factor $2/(2g+n-2)$ is equal to $\mathsf{ml}\,(v^2_{2g+n-2})/\mathsf{cp}\,(v^2_{2g+n-2})$.

Consider the summand of the second sum in the right hand side of this equation corresponding to i'. According to this summand we add to our graph the vertex v^2_{2g+n-2} with two sons, $v^0_{i'}$ and v^2_{2g+n-1}. We put $\mathsf{cp}\,(v^2_{2g+n-2}) = 2g+n-2$ and we have $\mathsf{ml}\,(v^2_{2g+n-2}) = 3$. Therefore, the factor $3/(2g+n-2)$ is equal to $\mathsf{ml}\,(v^2_{2g+n-2})/\mathsf{cp}\,(v^2_{2g+n-2})$.

Consider the summand of the second sum in the right hand side of this equation corresponding to $W_{g-1}(\eta_2\eta_1^{n-2})$. According to this summand we add to our graph the vertex v^1_{2g+n-2} with one sons, v^2_{2g+n-1}. We put $\mathsf{cp}\,(v^1_{2g+n-2}) = 2g+n-2$ and we have $\mathsf{ml}\,(v^1_{2g+n-2}) = 2$. Therefore, the factor $(2^3-2)/12(2g+n-2)$ is equal to $\mathsf{ml}\,(v^1_{2g+n-2})/\mathsf{cp}\,(v^1_{2g+n-2})$.

Continue with this procedure one obtain the same representation of $W_g(\eta_1^n)$ as we have described. The missed factor $1/n$ gives the contribution $1/n^{g+n-1}$; the same factor we have in our definition of $N(\Gamma)$.

4. Proofs of Theorems 2.1-2.3

4.1. Proof of Theorem 2.3. This Theorem consists of two formulas:

$$W_1(\eta_{a_1}) = \frac{a_1^2-1}{24}; \qquad W_0(\eta_{a_1}\eta_{a_2}) = 1. \tag{4.1}$$

Let us start with the second one. We have $W_0(\eta_{a_1}\eta_{a_2}) = \int_{V_0(a_1,a_2)} 1$. Since $V_0(a_1,a_2) = \overline{\mathcal{M}}_{0,3}$, it follows the required formula.

Consider now the first formula. It could be rewritten as

$$\int_{V_1(a_1)} \lambda_1 = \frac{a_1^2-1}{24}. \tag{4.2}$$

Note that $\lambda_1|_{V_1(a_1)} = \psi_1|_{V_1(a_1)}$. The integral $\int_{V_1(a_1)}\psi_1$ equals

$$\frac{1}{4a_1}\sum_{i=1}^{a_1-1} i(a_1-i)\int_{V_0(i,a_1-i)} 1 \tag{4.3}$$

(this follows from [**7**], Theorem 12.2). Since $W_0(i,a_1-i) = \int_{V_0(i,a_1-i)} 1 = 1$ and $\sum_{i=1}^{a_1-1} i(a_1-i) = (a_1^3-a_1)/6$, it follows that

$$\int_{V_1(a_1)}\lambda_1 = \int_{V_1(a_1)}\psi_1 = \frac{a_1^2-1}{24}. \tag{4.4}$$

4.2. Ionel Lemma. In this subsection we recall the Ionel Lemma, which is the main tool in the foregoing argument. For a more detailed explanation of this technique, see [**5, 7**].

Consider the space $\widehat{H}$ of admissible coverings of genus g, of degree n, with m critical values, and with partitions $A_1,\dots,A_m$, $A_i = (a_1^i,\dots,a_{l_i}^i)$, over these critical values. Of course, for any i, $\sum_{j=1}^{l_i} a_j^i = n$, and $\sum_{i=1}^m\sum_{j=1}^{l_i}(a_j^i-1) = nm - \sum_{i=1}^m l_i = 2g+2n-2$. Moreover, we consider admissible coverings with all marked preimages of all critical values.

There are two mapping of the space $\widehat{H}$. The first one, $\mathsf{st}:\widehat{H}\to\overline{\mathcal{M}}_{g,\sum_{i=1}^m l_i}$, takes an admissible covering to its source curve with marked preimages of critical values. The second mapping, $\mathsf{ll}:\widehat{H}\to\overline{\mathcal{M}}_{0,m}$, takes an admissible covering to its target curve with marked critical values.

Consider an admissible covering $f\in\widehat{H}$. Let

$$\mathsf{st}\,(f) = (C_g, x_1^1,\dots,x_{l_1}^1,\dots,x_1^m,\dots,x_{l_m}^m), \quad \mathsf{ll}\,(f) = (C_0,z_1,\dots,z_m).$$

We choose our notations to make x_j^i be a preimage of x_i of multiplicity a_j^i w. r. t. the covering f.

Let $\psi(x_j^i)$ be the first Chern class of the line bundle over $\overline{\mathcal{M}}_{g,\sum_{i=1}^m l_i}$, whose fiber at a moduli point $(C_g,x_1^1,\dots,x_{l_m}^m)$ is equal to $T^*_{x_j^i}C_g$. Let $\psi(z_i)$ be the first Chern class of the line bundle over $\overline{\mathcal{M}}_{o,m}$, whose fiber at a moduli point $(C_0,z_1,\dots,z_m)$ is equal to $T^*_{z_i}C_g$.

Lemma 4.1. (Ionel Lemma, [**5**]) *In cohomology ring of $\widehat{H}$ we have:*

$$a_j^i\mathsf{st}^*\psi(x_j^i) = \mathsf{ll}^*\psi(z_i). \tag{4.5}$$

4.3. Proof of Theorem 2.2.

4.3.1. Consider the space $V_g(a_1,\dots,a_n)$. Let $\widehat{H}$ be the space of admissible coverings of genus g, of degree $N=a_1+\dots+a_n$, with $m=2g+n+1$ critical values, and with partitions $A_1=(N)$, $A_2=(a_1,\dots,a_n)$, $A_3=\dots=A_m=(2,1,\dots,1)$ over these critical values.

In the foregoing we use the notations from the previous subsection.

Consider the projection $\pi\colon \overline{\mathcal{M}}_{g,\sum_{i=1}^m l_i}\to\overline{\mathcal{M}}_{g,1+n}$, forgetting all marked points except for $x_1^1,x_1^2,\dots,x_n^2$. Note that $\pi\circ\mathsf{st}\,(\widehat{H})=V_g(a_1,\dots,a_n)$. Moreover, $\pi_*\mathsf{st}\,_*[\widehat{H}]=(m-2)![V_g(a_1,\dots,a_n)]$.

We know that $\psi(x_1^1)|_{\mathsf{st}\,(\widehat{H})}=\pi^*\psi(x_1^1)$ (this is proved in [**7**]). Therefore,

$$(m-2)!\psi(x_1^1)|_{V_g(a_1,\dots,a_n)}=\pi_*\mathsf{st}\,_*\mathsf{st}\,^*\psi(x_1^1). \tag{4.6}$$

Since $N\mathsf{st}\,^*\pi^*\psi(x_1^1)=\mathsf{ll}\,^*\psi(z_1)$ (this is Ionel Lemma), it follows that

$$(m-2)!\psi(x_1^1)|_{V_g(a_1,\dots,a_n)}=\frac{1}{N}\pi_*\mathsf{st}\,_*\mathsf{ll}\,^*\psi(z_1). \tag{4.7}$$

So, we are to take a divisor dual to $\psi(z_1)$ on $\overline{\mathcal{M}}_{0,m}$, then we get its preimage in $\widehat{H}$, and the mapping $\pi\circ\mathsf{st}$ takes this preimage to the divisor in $V_g(a_1,\dots,a_n)$ dual to $\psi(x_1^1)|_{V_g(a_1,\dots,a_n)}$.

Note that $\psi(z_1)$ on $\overline{\mathcal{M}}_{0,m}$ is dual to the divisor, whose generic point is represented by a two-component curve such that z_1 lie on one component and z_2 and z_3 lie on the other component.

4.3.2. Consider $\psi(x_1^1)^{g+k-2}\lambda_g\cdot[V_g(a_1,\dots,a_k)]$. If $g+k-2=0$, then we have one of the cases considered in Theorem 2.3. Suppose that $g+k-2>0$. Then we have

$$\begin{aligned}\psi(x_1^1)^{g+k-2}\lambda_g\cdot[V_g(a_1,\dots,a_k)]=&\\ \frac{1}{N(m-2)!}\psi(x_1^1)^{g+k-3}&\lambda_g\cdot\pi_*\mathsf{st}\,_*\mathsf{ll}\,^*(\psi(z_1)\cdot[\overline{\mathcal{M}}_{0,m}])\end{aligned} \tag{4.8}$$

From dimensional conditions and since λ_g restricted to the divisor of irreducible self-intersecting curves equals zero, it follows that only two types of divisors contribute to $\psi(x_1^1)^{g+k-3}\lambda_g\cdot\pi_*\mathsf{st}\,_*\mathsf{ll}\,^*(\psi(z_1)\cdot\overline{\mathcal{M}}_{0,m})$. (We skip here some argument on the boundary behaviour of the space $\widehat{H}$. We refer to [**7**] for details.)

A divisor $D_{i,j}$ of the first type consists of two-compenent curves $(C,x_1^1,x_1^2,\dots,x_n^2)\in V_g(a_1,\dots,a_n)$ such that one component of C has genus zero and contains only x_i^2 and x_j^2. Here pairs (i,j) enumerate such divisors. Obviously, the restriction of $\psi(x_1^1)^{g+k-3}\lambda_g$ to the divisor $D_{i,j}$ is equal to the integral of this class over

$$V_g(a_1,\dots,\hat{a}_i,\dots,\hat{a}_j,\dots,a_n,a_i+a_j).$$

The multiplicity of the mapping $\pi \circ \mathsf{st}$ over $D_{i,j}$ equals $(m-3)!$. The multiplicity of the mapping ll at $(\pi \circ \mathsf{st})^{-1}(D_{i,j})$ equals $a_i + a_j$. Thus we obtain that the coefficient of the correspoding summand in Equation 2.3 equals

$$\frac{(a_i+a_j)(m-3)!}{N(m-2)!} = \frac{(a_i+a_j)}{(2g+n-1)(\sum_{i=1}^n a_i)}. \tag{4.9}$$

A divisor D_i of the second type consists of two-component curves $(C, x_1^1, x_1^2, \dots, x_n^2) \in V_g(a_1, \dots, a_n)$ such that one component of C has genus one and contains only x_i. Obviously, the restriction of $\psi(x_1^1)^{g+k-3}\lambda_g$ to the divisor $D_{i,j}$ is equal to the integral of $\psi(x_1^1)^{g+k-3}\lambda_{g-1}$ over $V_{g-1}(a_1, \dots, a_n)$ multiplied by the integral of λ_1 over $V_1(a_i)$. It follows from Theorem 2.3 that last factor equals $(a_i^2-1)/24$.

The multiplicity of the mapping $\pi \circ \mathsf{st}$ over D_i equals $2 \cdot (m-3)!$. The multiplicity of the mapping ll at $(\pi \circ \mathsf{st})^{-1}(D_{i,j})$ equals a_i. Thus we obtain that the coefficient of the correspoding summand in Equation 2.3 equals

$$a_i \cdot \frac{(a_i^2-1)}{24} \cdot \frac{2(m-3)!}{N(m-2)!} = \frac{a_i^3 - a_i}{12(2g+n-1)(\sum_{i=1}^n a_i)}. \tag{4.10}$$

This concludes the proof.

4.4. Proof of Theorem 2.1.

4.4.1. Let us fix g and $a_1, \dots, a_n$. By V_g^i denote

$$V_g^i := V_g(a_1, \dots, a_n \underbrace{1, \dots, 1}_{i}). \tag{4.11}$$

Recall that V_g^i is the subspace of $\overline{\mathcal{M}}_{g,n+i+1}$ consisting of curves $(C, x_1, \dots, x_{n+i+1})$ such that

$$-(N+i)x_1 + a_1x_2 + \dots + a_nx_{n+1} + x_{n+2} + \dots + x_{n+i+1}$$

is the divisor of a meromorphic function.

Consider the mappings $\pi_{j,i}\colon V_g^j \to \overline{\mathcal{M}}_{g,n+j+1-i}$ forgetting the marked points $x_{n+j+2-i}, \dots, x_{n+j+1}$. Note that $\pi_{j,i,*}[V_g^j] = i![\pi_{i,*}(V_g^j)]$, $\pi_{j,i}(V_g^j) = \pi_{j+1,i+1}(V_g^{j+1})$, and $\pi_{g,g}(V_g^g) = \overline{\mathcal{M}}_{g,n+1}$.

Recall that our goal is to get an expression of $\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2}\lambda_g = \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{2g+n-2}\lambda_g$. We shall do this in g steps.

4.4.2. Consider the projection $\sigma\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$ forgetting the last marked point. We have

$$\sigma^*(\psi_1)^{2g+n-2} = \psi_1^{2g+n-2} - \sigma^*(\psi_1)^{2g+n-3} \cdot D, \tag{4.12}$$

where D is the class of the divisor, whose generic point is represented by a two-component curve such that one component has genus zero and contains x_1 and x_{n+2} and the other component has genus g and contains all other points.

Since $\sigma^*\lambda = \lambda$, we have:

$$\begin{aligned} I := \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{2g+n-2}\lambda_g = \\ \int_{\pi_{g,g}(V_g^g)} \psi_1^{2g+n-2}\lambda_g = \frac{1}{g}\int_{\pi_{g,g-1}(V_g^g)} \sigma^*(\psi_1)^{2g+n-2}\lambda_g \end{aligned} \tag{4.13}$$

Note that

$$\sigma(D \cap \pi_{g,g-1}(V_g^g)) = \pi_{g-1,g-1}(V_g^{g-1}). \tag{4.14}$$

Therefore,

$$I = \frac{1}{g}\int_{\pi_{g,g-1}(V_g^g)} \psi_1^{2g+n-2}\lambda_g - \frac{1}{g}\int_{\pi_{g-1,g-1}(V_g^{g-1})} \psi_1^{2g+n-3}\lambda_g. \tag{4.15}$$

4.4.3. Applying the same argument to the right hand side of Equation 4.15 we get:

$$\begin{aligned} \int_{\pi_{g,g-1}(V_g^g)} \psi_1^{2g+n-2}\lambda_g =& \frac{1}{g-1}\int_{\pi_{g,g-2}(V_g^g)} \psi_1^{2g+n-2}\lambda_g \\ &- \frac{1}{g-1}\int_{\pi_{g-1,g-2}(V_g^{g-1})} \psi_1^{2g+n-3}\lambda_g \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \int_{\pi_{g-1,g-1}(V_g^{g-1})} \psi_1^{2g+n-3}\lambda_g =& \frac{1}{g-1}\int_{\pi_{g-1,g-2}(V_g^{g-1})} \psi_1^{2g+n-3}\lambda_g \\ &- \frac{1}{g-1}\int_{\pi_{g-2,g-2}(V_g^{g-2})} \psi_1^{2g+n-4}\lambda_g \end{aligned} \tag{4.17}$$

Thus we have:

$$\begin{aligned} g(g-1)I = \int_{\pi_{g,g-2}(V_g^g)} \psi_1^{2g+n-2}\lambda_g - 2\int_{\pi_{g-1,g-2}(V_g^{g-1})} \psi_1^{2g+n-3}\lambda_g \\ + \int_{\pi_{g-2,g-2}(V_g^{g-2})} \psi_1^{2g+n-4}\lambda_g. \end{aligned} \tag{4.18}$$

Continue with this procedure we obtain:

$$g!I = \binom{g}{0}\int_{\pi_{g,0}(V_g^g)} \psi_1^{2g+n-2}\lambda_g - \binom{g}{1}\int_{\pi_{g-1,0}(V_g^{g-1})} \psi_1^{2g+n-3}\lambda_g + \cdots + (-1)^g\binom{g}{g}\int_{\pi_{0,0}(V_g^0)} \psi_1^{g+n-2}\lambda_g. \tag{4.19}$$

Since $\pi_{i,0}$ are identical mappings, this equation is exactly the statement of Theorem 2.1.

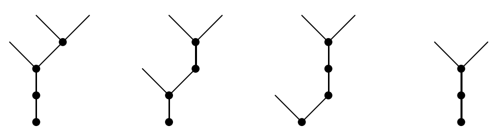
Appendix A. Some calculations in genus 2

Here we give some calculations checking independently in a special case Corollary 3.2 and therefore the whole algorithm for computing Hodge integrals.

Let us put $g = 2$, $n = 1$. Recall that $B_4 = -1/30$. So, we would like to check that

$$S_{2,3} - 2S_{2,2} = \frac{7\cdot 2}{8\cdot 24\cdot 30} = \frac{7}{2^6\cdot 3^2\cdot 5} \tag{A.1}$$

We have already studied in examples all types of graphs contributing to $S_{2,3}$ and $S_{2,2}$:



The number $S_{2,3}$ is determined by the first three types of graphs. These graphs have 3 possible mappings nm . The mappings cp and ml for these graphs are defined above. Thus we have that the first type of graphs contributes

$$3\cdot\frac{1}{3^4}\frac{2\cdot 3}{2\cdot 4\cdot 5\cdot 6}\left(\frac{24}{12}\right)^2 = \frac{1}{2\cdot 3^3\cdot 5}, \tag{A.2}$$

the second type of graphs contributes

$$3\cdot\frac{1}{3^4}\frac{2\cdot 3}{2\cdot 3\cdot 5\cdot 6}\frac{6\cdot 24}{12^2} = \frac{1}{2\cdot 3^4\cdot 5}, \tag{A.3}$$

and the third type of graphs contributes

$$3\cdot\frac{1}{3^4}\frac{2\cdot 3}{1\cdot 3\cdot 5\cdot 6}\left(\frac{6}{12}\right)^2 = \frac{1}{2^2\cdot 3^4\cdot 5}. \tag{A.4}$$

Therefore,

$$S_{2,3} = \frac{1}{2^2\cdot 3^2\cdot 5}. \tag{A.5}$$

The number $S_{2,2}$ is determined by the fourth type of graph. This graph has the unique possible mapping nm. The mappings cp and ml for this graph are also defined above. Thus we have

$$S_{2,2} = \frac{1}{2^3}\frac{2}{2\cdot 4\cdot 5}\left(\frac{6}{12}\right)^2 = \frac{1}{2^7\cdot 5}. \tag{A.6}$$

Therefore,

$$S_{2,3} - 2S_{2,2} = \frac{1}{2^2\cdot 3^2\cdot 5} - \frac{1}{2^6\cdot 5} = \frac{7}{2^6\cdot 3^2\cdot 5}. \tag{A.7}$$

The same equality is given by Corollary 3.2.

Appendix B. Genuz 0 case

In genus zero we have the most simple and beautiful combinatorics. Moreover, in this case our argument with intersections on the moduli spaces of curves could be considered as a purely combinatorial argument.

B.1. Genus zero combinatorial statement. Let us fix $g \geq 0$ and $n \geq 1$. We describe here trees corresponding to the number $W_g(\eta_1^n)$.

We consider rooted trees. Each vertex has either 2 sons or no sons. We have exactly n vertices with no sons (leaves) an $n-1$ vertices with two sons.

By V_0 denote the set of leaves. An item of the construction is a one-to-one correspondence $\mathsf{nm} : V_0 \to \{1, \dots, n\}$.

By V_2 denote the set of vertices with two sons. Consider a one-to-one map $\mathsf{cp} : V_2 \to \{1, \dots, n-1\}$ satifying the following property: If vertex v is a 'descendant' of vertex v', then $\mathsf{cp}(v) > \mathsf{cp}(v')$.

By a decorated n-tree we call all this data, i. e. a rooted tree with mappings nm and cp.

Consider a decorated n-tree. Let us define one more function $\mathsf{ml} : V_2 \to \{1, \dots, n\}$. The mapping ml takes a vertex to the number of its descendants in V_0.

Let Γ be a decorated tree. By $N(\Gamma)$ denote the number

$$N(\Gamma) := \frac{1}{n^{(n-1)}}\prod_{v\in V_2}\frac{\mathsf{ml}(v)}{\mathsf{cp}(v)} \tag{B.1}$$

The special case of Theorem 3.1 looks like follows:

THEOREM B.1. *For any n we have*

$$\sum_{\Gamma} N(\Gamma) = 1, \tag{B.2}$$

where the sum is taken over all decorated n-trees Γ.

B.2. Proof of Theorem B.1. As we have already shown in Section 3, $\sum_\Gamma N(\Gamma)$ equals the number $W_0(\eta_1^n)$, where the numbers $W_0(\prod_{i=1}^k \eta_{a_i})$ are defined by the recursion relation

$$(\sum_{i=1}^k a_i)(k-1)W_0(\prod_{i=1}^k \eta_{a_i}) = \sum_{k<l}(a_k+a_l)W_0(\eta_{a_k+a_l}\prod_{i\neq k,l}\eta_{a_i}), \tag{B.3}$$

and initial data

$$W_0(\eta_{a_1}\eta_{a_2}) = 1 \tag{B.4}$$

(we give here the special case of this relation for genus 0).

Note that the numbers $W_0(\prod_{i=1}^k \eta_{a_i})$ are uniquely determined by this relation. Note also that $W_0(\prod_{i=1}^k \eta_{a_i}) = 1$ obviously satisfy this relation and the initial data. So, for any $a_1,\dots,a_k$ $W_0(\prod_{i=1}^k \eta_{a_i}) = 1$. In particular, $W_0(\eta_1^n) = 1$.

Appendix C. The other Hodge integrals

In this section we give an algorithm to calculate any Hodge integral

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2-i}\lambda_i. \tag{C.1}$$

C.1. Algorithm. By $W_g^i(\eta_{a_1}\dots\eta_{a_n})$ denote the intersection number

$$W_g^i(\eta_{a_1}\dots\eta_{a_n}) := \int_{V_g(a_1,\dots,a_n)} \psi_1^{2g+n-2-i}\lambda_i \tag{C.2}$$

The first step of the algorithm is just the same as in the λ_g-case.

THEOREM C.1. *For arbitrary positive integers $a_1,\dots,a_n$ we have*

(C.3)

$$(-1)^g g!\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2-i}\lambda_i = \binom{g}{0}W_g^i(\prod_{i=1}^n\eta_{a_i}) - \binom{g}{1}W_g^i(\eta_1\prod_{i=1}^n\eta_{a_i}) + \binom{g}{2}W_g^i(\eta_1^2\prod_{i=1}^n\eta_{a_i}) - \dots + (-1)^g\binom{g}{g}W_g^i(\eta_1^g\prod_{i=1}^n\eta_{a_i})$$

The recursion relation for the numbers $W_g^i(\prod_{i=1}^n \eta_{a_i})$ looks like follows:

THEOREM C.2. *If $2g+n-2-i>0$, then*

(C.4)

$$(\sum_{i=1}^{n} a_i)(2g+n-1)W_g^i(\prod_{i=1}^{n}\eta_{a_i}) = \sum_{k<l}(a_k+a_l)W_g^i(\eta_{a_k+a_l}\prod_{i\neq k,l}\eta_{a_i})$$

$$+\sum_{k=1}^{n}\frac{a_k^3-a_k}{12}W_{g-1}^{i-1}(\prod_{i=1}^{n}\eta_{a_i})+\frac{1}{2}\sum_{k=1}^{n}\sum_{a_k'+a_k''=a_k}a_k'a_k''W_{g-1}^i(\eta_{a_k'}\eta_{a_k''}\prod_{i\neq k}\eta_{a_i}).$$

The initial values are just the same as in the λ_g-case.

THEOREM C.3. *If $i>g$ then $W_g^i(\eta_{a_1}\dots\eta_{a_n})=0$. Besides,*

$$W_1^1(\eta_{a_1})=\frac{a_1^2-1}{24};\qquad W_0^0(\eta_{a_1}\eta_{a_2})=1. \tag{C.5}$$

Proofs of these Theorems are very similiar to the proofs of Theorems 2.1-2.3. So we skip the proofs.

C.2. Some calculations. Let us apply our algorithm to compute the integral $\int_{\overline{\mathcal{M}}_{2,1}}\psi_1^3\lambda_1$. Using Theorem C.1, we get

$$\int_{\overline{\mathcal{M}}_{2,1}}\psi_1^3\lambda_1=\frac{1}{2}W_2^1(\eta_1^3)-W_2^1(\eta_1^2). \tag{C.6}$$

Applying many times Theorems C.2 and C.3, we obtain

$$W_2^1(\eta_1^3)=\frac{1}{120};\qquad W_2^1(\eta_1^2)=\frac{1}{480}. \tag{C.7}$$

Therefore,

$$\int_{\overline{\mathcal{M}}_{2,1}}\psi_1^3\lambda_1=\frac{1}{240}-\frac{1}{480}=\frac{1}{480}. \tag{C.8}$$

The same number is given by Equation 1.2.

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